The Fourier Convergence Theorem

- We know that if a Fourier series

$$\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{L}+b_m\sin\frac{m\pi x}{L}\right)$$

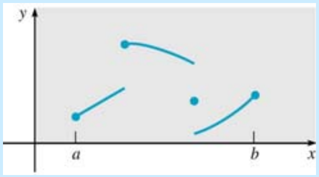
converges and thereby defines a function *f*, then *f* is periodic with period 2*L*, with the coefficients a_m and b_m given by

$$a_m=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{m\pi x}{L}dx,\quad b_m=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{m\pi x}{L}dx$$

- In this section we begin with a periodic function *f* of period 2*L* that is integrable on [-*L*, *L*]. We compute a_m and b_m using the formulas above and construct the associated Fourier series.
- The question is whether this series converges for each *x*, and if so, whether its sum is *f*(*x*).

Fourier Series Representation of Functions

- To guarantee convergence of a Fourier series to the function from which its coefficients were computed, it is essential to place additional conditions on the function.
- From a practical point of view, such conditions should be broad enough to cover all situations of interest, yet simple enough to be easily checked for particular functions.
- Recall the definition of a piecewise continuous function on the next slide.



Piecewise Continuous Functions

- A function f is **piecewise continuous** on an interval $[a, b]$ if this interval can be partitioned by a finite number of points $a = x_0 < x_1 < \ldots < x_n = b$ such that
  - (1) f is continuous on each (x_k, x_{k+1})
  - (2) $\left|\lim_{x \to t_k^+} f(x)\right| < \infty, \quad k = 0, \ldots, n-1$
  - (3) $\left|\lim_{x \to x_{k+1}^-} f(x)\right| < \infty, \quad k = 1, \ldots, n$

- The notation $f(c+)$ denotes the limit of $f(x)$ as $x \to c$ from the right, and $f(c-)$ denotes the limit of $f(x)$ as $x \to c$ from the left.
- It is not essential that the function be defined at the partition points x_k, nor is it essential that the interval $[a, b]$ be closed.

Theorem :Dirichlet's Condition

- Suppose that f and f' are piecewise continuous on $[-L, L)$.
- Further, suppose that f is defined outside $[-L, L)$ so that it is periodic single valued, finite with period $2L$ and at most finite number of maxima and minima.
- The f has a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$$

where

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx, \quad b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx$$

- The Fourier series converges to $f(x)$ at all points x where f is continuous, and to $[f(x+) + f(x-)]/2$ at all points x where f is discontinuous.

Theorem : Discussion

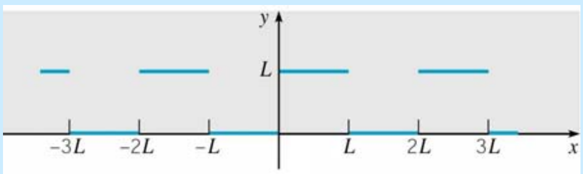
- Note that the Fourier series converges to the average of $f(x+)$ and $f(x-)$ at the discontinuities of f.
- The conditions given in this theorem are only sufficient for the convergence of a Fourier series; they are not necessary. Nor are they the most general sufficient conditions possible.
- Functions that are not included in the theorem are primarily those with infinite discontinuities in [$-L$, L), such as $1/x^2$.
- A Fourier series may converge to a function that is not differentiable or continuous, even though each term in the series is continuous and infinitely differentiable.
- The next example illustrates this.

Example 1: Square Wave (1 of 8)

- Consider the function below.

$$f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L \end{cases}, \qquad f(x+2L) = f(x)$$

- We temporarily leave open the definition of f at $x = 0$ and $x = \pm L$, except to say that its value must be finite.
- This function represents a square wave, and is periodic with period $T = 2L$. See graph of f below.

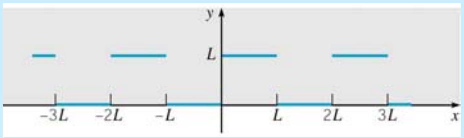


Example 1: Square Wave (2 of 8)

- Recall that for our function f,

$$f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L \end{cases}, \qquad f(x+2L) = f(x)$$

- The interval [-L, L) can be partitioned to give two open subintervals (-L, 0) and (0, L).
- On (0, L), $f(x) = L$ and $f'(x) = 0$. Thus f and f' are continuous and have finite limits as $x \to 0$ from right and $x \to L$ from left.
- Similarly on (-L, 0). Thus f and f' are piecewise continuous on [-L, L].



Example 1: Coefficients (3 of 8)

- First, we find a_0:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L L\,dx = L$$

- Then for a_m, $m = 1, 2, \ldots$, we have

$$a_m = \frac{1}{L}\int_0^L L\cos\frac{m\pi x}{L}\,dx = \frac{L}{m\pi}\sin\frac{m\pi x}{L}\bigg|_0^L = 0, \ \ m \neq 0$$

- Similarly, for $b_m = 0$, $m = 1, 2, \ldots$,

$$b_m = \frac{1}{L}\int_0^L L\sin\frac{m\pi x}{L}\,dx = \frac{L}{m\pi}\cos\frac{m\pi x}{L}\bigg|_0^L = \begin{cases} 2L/m\pi, & m \text{ odd} \\ 0, & m \text{ even,} \end{cases}$$

Example 1: Fourier Expansion (4 of 8)

- Thus $a_m = 0$, $m = 1, 2, \ldots$, and

$$a_0 = L,\ b_m = \begin{cases} 2L/m\pi, & m \text{ odd} \\ 0, & m \text{ even}, \end{cases}$$

- Then

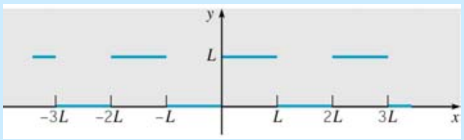
$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right) \\ &= \frac{L}{2} + \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \ldots\right) \\ &= \frac{L}{2} + \frac{2L}{\pi}\sum_{m=1,3,5,\ldots}^{\infty}\frac{\sin(m\pi x/L)}{L} \\ &= \frac{L}{2} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x/L}{2n-1} \end{aligned}$$

Example 1: Theorem (5 of 8)

- Thus

$$f(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x/L}{2n-1}$$

- Now f is continuous on $(-nL, 0)$ and $(0, nL)$, hence the Fourier series converges to $f(x)$ on these intervals.
- At the points $x = 0, \pm nL$ where f is discontinuous, all terms after the first vanish, and the sum is $L/2 = [f(x+) + f(x-)]/2$.
- Thus we may choose to define $f(x)$ to be $L/2$ at these points of discontinuity, for then series will converge to f at these points.

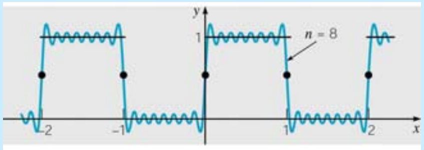


Example 1: Gibbs Phenomena (6 of 8)

- Consider the partial sum

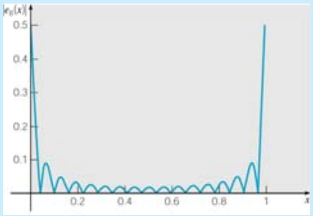
$$s_n(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{k=1}^{n}\frac{\sin(2k-1)\pi x / L}{2k-1}$$

- The graphs of $s_8(x)$ and f are given below for $L = 1$.
- The partial sums appear to converge to f at points of continuity while they tend to overshoot f near points of discontinuity.
- This behavior is typical of Fourier series at points of discontinuity and is known as Gibbs phenomena.



Example 1: Errors (7 of 8)

- To investigate the convergence in more detail, we consider the error function $e_n(x) = f(x) - s_n(x)$.
- Given below is a graph of $|e_8(x)|$ and $L = 1$. The least upper bound of $|e_8(x)|$ is 0.5, and is approached as $x \to 0$ and $x \to 1$.
- As n increases, the error decreases on (0, 1), where f is continuous, but the least upper bound for the error does not diminish with increasing n.
- Thus we cannot uniformly reduce the error throughout the interval by increasing the number of terms.



Example 1: Speed of Convergence (8 of 8)

- Note that in our Fourier series,

$$f(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x / L}{2n-1}$$

the coefficients are proportional to $1/(2n-1)$.

- Thus this series converges more slowly whose coefficients are proportional to $1/(2n-1)^2$.