Fourier Series and Fourier Transform

What is a Fourier series ?

Let $f(x)$ be a PERIODIC function of period 2π, and $f(x)$ satisfies some "commonly practical" conditions, then, $f(x)$ can be represented by a trigonometric series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad \text{and}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \quad n = 1, 2, 3, \ldots\text{and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx, \quad n = 1, 2, 3, \ldots$$

and the a_0, a_n's and b_n's are the so-called Fourier coefficients.

Derivation of the Fourier Coefficients

Let $f(x)$ be a PERIODIC function of period 2π, and $f(x)$ satisfies some "commonly practical" conditions, then, $f(x)$ can be represented by a trigonometric series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Note that if we integrate both sides of (*) from $x = -\pi$ to $x = \pi$, we have:

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} (a_n \int_{-\pi}^{\pi} \cos nxdx + b_n \int_{-\pi}^{\pi} \sin nxdx)$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} f(x)dx = 2\pi a_0 + \sum_{n=1}^{\infty} (a_n \left.\frac{\sin nx}{n}\right]_{-\pi}^{\pi} + b_n \left.\frac{-\cos nx}{n}\right]_{-\pi}^{\pi}) = 2\pi a_0$$

$$\Rightarrow \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

Derivation of the Fourier Coefficients (cont'd)

Now that $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$................(*)

If we multiply both sides of (*) with $\cos mx$ and then integrate both sides from $x = -\pi$ to $x = \pi$, we have:

$$\int_{-\pi}^{\pi} f(x)\cos mx dx = a_0 \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx\right) \text{....(**)}$$

But: $\int_{-\pi}^{\pi} \cos mx dx = 0$ and

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx = \begin{cases} 0 & \text{for } n \neq m \\ \pi & \text{for } n = m \end{cases} \quad \text{and}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x dx = 0$$

Thus, (**)

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx dx = a_0 \cdot 0 + a_m \pi + 0$$

$$\Rightarrow a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx, \qquad m = 1, 2, \ldots$$

Derivation of the Fourier Coefficients (cont'd)

Now that $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.................(*)

If we multiply both sides of (*) with $\sin mx$ and then integrate both sides from $x = -\pi$ to $x = \pi$, we have:

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = a_0 \int_{-\pi}^{\pi}\sin mxdx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi}\cos nx \sin mxdx + b_n \int_{-\pi}^{\pi}\sin nx \sin mxdx\right)....(**)$$

But: $\int_{-\pi}^{\pi}\sin mxdx = 0$ and

$$\int_{-\pi}^{\pi}\cos nx \sin mxdx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(m+n)xdx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(m-n)xdx = 0 \quad \text{and}$$

$$\int_{-\pi}^{\pi}\sin nx \sin mxdx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx - \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)xdx = \begin{cases} 0 & \text{for } n \neq m \\ \pi & \text{for } n = m \end{cases} \quad \text{and}$$

Thus, (**)

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin mxdx = a_0 \cdot 0 + 0 + b_m \pi$$

$$\Rightarrow b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mxdx, \quad m = 1, 2, \ldots$$

Example of Fourier Series

Consider $f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases}$ and $f(x+2\pi) = f(x).$

Consider the Fourier series representation of

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}[\int_{-\pi}^{0} -kdx + \int_{0}^{\pi} kdx] = \frac{k}{2\pi}(\pi - \pi) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}[\int_{-\pi}^{0} -k\cos nxdx + \int_{0}^{\pi} k\cos nxdx] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}[\int_{-\pi}^{0} -k\sin nxdx + \int_{0}^{\pi} k\sin nxdx]$$

$$= \frac{k}{\pi}(\frac{\cos nx}{n}\Big]_{-\pi}^{0} - \frac{\cos nx}{n}\Big]_{0}^{\pi}) = \frac{k}{\pi n}[1-(-1)^n - (-1)^n + 1)] = \frac{2k}{\pi n}[1-(-1)^n]$$

Therefore, $b_n = \begin{cases} 0 & \text{for } n = 2,4,6,\ldots \\ \dfrac{4k}{n\pi} & \text{for } n = 1,3,5,\ldots \end{cases}$

Thus, $f(x) = \dfrac{4k}{\pi}(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x + \ldots)$

Fourier series for Odd functions

Let $f(x)$ be a PERIODIC function of period 2π, and $f(x)$ satisfies some "commonly practical" conditions, then, $f(x)$ can be represented by a

Fourier series: $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

If $f(x)$ is an odd function, i.e. $f(-x) = -f(x)$ for all x, then:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx] = \frac{1}{2\pi}[\int_{x^*=\pi}^{x^*=0} f(-x^*)d(-x^*) + \int_{0}^{\pi} f(x)dx]$$

$$= \frac{1}{2\pi}[\int_{x^*=0}^{x^*=\pi} f(-x^*)dx^* + \int_{0}^{\pi} f(x)dx] = \frac{1}{2\pi}[-\int_{x^*=0}^{x^*=\pi} f(x^*)dx^* + \int_{0}^{\pi} f(x)dx] = 0$$

Also,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}[\int_{-\pi}^{0} f(x)\cos nx dx + \int_{0}^{\pi} f(x)\cos nx dx]$$

$$= \frac{1}{\pi}[\int_{x^*=\pi}^{x^*=0} f(-x^*)\cos(-nx^*)d(-x^*) + \int_{0}^{\pi} f(x)\cos nx dx]$$

$$= \frac{1}{\pi}[-\int_{x^*=\pi}^{x^*=0} -f(x^*)\cos(nx^*)dx^* + \int_{0}^{\pi} f(x)\cos nx dx] = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Thus, if $f(x)$ is an odd periodic function with period of 2π, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Fourier series for Even functions

Let $f(x)$ be a PERIODIC function of period 2π, and $f(x)$ satisfies some "commonly practical" conditions, then, $f(x)$ can be represented by a

Fourier series: $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

If $f(x)$ is an even function, i.e. $f(-x) = f(x)$ for all x, then:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}[\int_{-\pi}^{0} f(x)\sin nx dx + \int_{0}^{\pi} f(x)\sin nx dx]$$

$$= \frac{1}{\pi}[\int_{x^*=\pi}^{x^*=0} f(-x^*)\sin(-nx^*)d(-x^*) + \int_{0}^{\pi} f(x)\sin nx dx]$$

$$= \frac{1}{\pi}[-\int_{x^*=\pi}^{x^*=0} f(x^*)(-1)\sin(nx^*)dx^* + \int_{0}^{\pi} f(x)\cos nx dx] = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Thus, if $f(x)$ is an even periodic function with period of 2π , then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Representation by a Fourier series

Theorem:

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \leq x \leq \pi$, and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series (*) of $f(x)$ is convergent and its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Functions of any period p=2L

Let $f(x)$ be a PERIODIC function of period $2L$, and $f(x)$ satisfies some "commonly practical" conditions, then, $f(x)$ has a Fourier series given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x)..................(***)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \quad \text{and}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots\text{and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Proof:

Substitute $v = \frac{\pi x}{L} \Leftrightarrow x = \frac{Lv}{\pi}$, then $x = \pm L$ corresponds to $v = \pm\pi$

thus, let $g(v) = f(x)$. Then:

$$g(v+2\pi) = f(\frac{L(v+2\pi)}{\pi}) = f(\frac{Lv}{\pi} + 2L) = f(x+2L) = f(x) = g(v)$$

$\Rightarrow g(v)$ has a period of 2π.

We can therefore expand $g(v)$ using the previous results of (*).

Functions of any period p=2L (Example)

Let $f(x)$ be a PERIODIC function of period $2L$, and $f(x)$ satisfies some "commonly practical" conditions, then, $f(x)$ has a Fourier series given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x) \text{..................(***)}$$

where $a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx,\ \ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx,\ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$

Find the Fourier series for $f(x) = \begin{cases} 0 \text{ if } -2 < x < -1 \\ k \quad \text{if } -1 < x < 1 \\ 0 \quad \text{if } 1 < x < 2 \end{cases}$ $period = 2L = 4 \Rightarrow L = 2$

Observe that $f(x)$ is an even function $\Rightarrow b_n\text{'}s = 0$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx = \frac{1}{4} \int_{-2}^{2} f(x) dx = \frac{1}{4} \int_{-1}^{1} k dx = \frac{k}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{2} \int_{-1}^{1} k \cos \frac{n\pi x}{2} dx = \frac{2k}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{2k}{n\pi} \quad \text{if } n = 1, 5, 9, \ldots \\ \frac{-2k}{n\pi} \quad \text{if } n = 3, 7, 11, \ldots \end{cases}$$

Thus, $f(x) = \frac{k}{2} + \frac{2k}{\pi} (\cos \frac{\pi}{2} x - \frac{1}{3} \cos \frac{3\pi}{2} x + \frac{1}{5} \cos \frac{5\pi}{2} x - \frac{1}{7} \cos \frac{7\pi}{2} x + \ldots)$

Half-range Expansion

Consider a function $f(x)$ which is only defined for the interval $0 \le x \le L$, then we can construct

$$f_1(x) = \begin{cases} f(x) \text{ for } 0 \le x \le L \\ -f(x) \text{ for } -L \le x \le 0 \end{cases} \text{ and } f_1(x+2L) = f_1(x)$$

Then, $f_1(x)$ is a periodic, ODD, function of period $2L$, and thus, we can obtain the Fourier series expansion for $f_1(x)$, i.e.

$$f_1(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\text{.............}(**)$$

where $\quad b_n = \frac{1}{L}\int_{-L}^{L} f_1(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$

Also that $\quad b_n = \frac{1}{L}[\int_{-L}^{0} f_1(x)\sin\frac{n\pi x}{L}dx + \int_{0}^{L} f_1(x)\sin\frac{n\pi x}{L}dx]$

$$= \frac{1}{L}[\int_{x^*=L}^{x^*=0} f_1(-x^*)\sin\frac{n\pi(-x^*)}{L}d(-x^*) + \int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx]$$

$$= \frac{1}{L}[\int_{x^*=L}^{x^*=0} -f_1(x^*)\sin\frac{n\pi(x^*)}{L}d(x^*) + \int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx]$$

$$= \frac{1}{L}[\int_{x^*=0}^{x^*=L} f_1(x^*)\sin\frac{n\pi(x^*)}{L}d(x^*) + \int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx]$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Once b_n's are determined, (**) can be used to represent $f(x)$ for $0 \le x \le L$ because $f(x) = f_1(x) =$ for $0 \le x \le L$.

Half-range Expansion (cont'd)

Alternatively, for the function $f(x)$ which is only defined for the interval $0 \le x \le L$, we can construct

$$f_2(x) = \begin{cases} f(x) \text{ for } 0 \le x \le L \\ f(-x) \text{ for } -L \le x \le 0 \end{cases} \text{ and } \; f_2(x+2L) = f_2(x)$$

Then, $f_2(x)$ is a periodic, EVEN, function of period $2L$, and thus, we can obtain the Fourier series expansion for $f_2(x)$, i.e.

$$f_2(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x ....(**)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_2(x)dx = \frac{1}{L}\int_0^L f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_2(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad n = 1, 2, 3, ...$$

Once the a_n's are determined, (**) can be used to represent $f(x)$ for $0 \le x \le L$ because $f(x) = f_2(x) =$ for $0 \le x \le L$.

Complex Fourier series

Consider the Fourier series representation for a PERIODIC function $f(x)$ of period 2π:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

It can be shown that $f(x)$ can also be represented in the following alternative form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

Complex Fourier series (cont'd)

Proof: Consider $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$....................(*)

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

Recall that $e^{inx} = \cos nx + i \sin nx;\quad e^{-inx} = \cos nx - i \sin nx$

$\Rightarrow \cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$ and $\sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right) = \frac{-i}{2}\left(e^{inx} - e^{-inx}\right)$

$\Rightarrow (a_n \cos nx + b_n \sin nx) = \frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) - i\frac{b_n}{2}\left(e^{inx} - e^{-inx}\right) = \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}$

Define $c_0 = a_0$; and $c_n = \frac{1}{2}(a_n - ib_n)$ and $k_n = \frac{1}{2}(a_n + ib_n)$; (*) becomes:

$$f(x) = c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + k_n e^{-inx}) \text{..................(**)}$$

Also that $c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)\cos nx - i \cdot f(x)\sin nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$

and $k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)\cos nx + i \cdot f(x)\sin nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$

Finally, by defining $c_{-n} = k_n$ and substitute k_n by c_{-n} in (**) to get:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

Complex Fourier series (cont'd)

For a periodic function $f(x)$ of period $=2\pi$,

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \qquad n=0,\pm1,\pm2,\ldots$$

is the so-called complex form of Fourier series or just complex Fourier series

For a periodic function $f(x)$ with period $= 2L$, the corresponding complex Fourier series is given by:

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$

where $$c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}dx, \qquad n=0,\pm1,\pm2,\ldots$$

Complex Fourier series: An Example

Consider a periodic function $f(x)$ of period $=2\pi$, where $f(x)=e^{x}$ for $-\pi<x<\pi$

Let $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,\quad n=0,\pm 1,\pm 2,\ldots$

then

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}e^{-inx}dx=\frac{1}{2\pi}\left.\frac{e^{(1-in)x}}{1-in}\right]_{-\pi}^{\pi}=\frac{1}{2\pi}\frac{e^{-inx}(e^{\pi}-e^{-\pi})}{1-in}$$

And since $e^{\pm in\pi}=\cos n\pi\pm\sin n\pi=\cos n\pi=(-1)^n$

and $\frac{1}{1-in}=\frac{1+in}{(1-in)(1+in)}=\frac{1+in}{1+n^2}$, and $\sinh\pi=\frac{e^{\pi}-e^{-\pi}}{2}$

$$\Rightarrow c_n=\frac{(-1)^n\sinh\pi(1+in)}{\pi(1+n^2)}\Rightarrow f(x)=e^{x}=\frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1+in)}{(1+n^2)}e^{inx},\qquad -\pi<x<\pi$$

Furthermore, $(1+in)e^{inx}=(1+in)(\cos nx+i\sin nx)=(\cos nx-n\sin nx)+i(n\cos nx+\sin nx)$

and $(1-in)e^{-inx}=(1-in)(\cos nx-i\sin nx)=(\cos nx-n\sin nx)-i(n\cos nx+\sin nx)$

$\Rightarrow (1+in)e^{inx}+(1-in)e^{-inx}=2(\cos nx-n\sin nx)$

Thus, $f(x)=e^{x}=\frac{\sinh\pi}{\pi}[1+\sum_{n=1}^{\infty}\frac{(-1)^n 2(\cos nx-n\sin nx)}{(1+n^2)}],\qquad -\pi<x<\pi$

Representation for Aperiodic function: Fourier Transform

Consider an APERIODIC function $f(x)$.

Define $F(i\omega)$ be the Fourier Transform of $f(x)$ as follows:

$$F(i\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

It can be shown that:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega x} d\omega$$

Let's define $\mathcal{F}\{\ \}$ as the Fourier transform operator, then

$$F(i\omega) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

also define $\mathcal{F}^{-1}\{\ \}$ as the Inverse Fourier transform operator, then

$$f(x) = \mathcal{F}^{-1}\{F(i\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega x} d\omega$$

Representation for Aperiodic function: Fourier Transform

Consider an APERIODIC function $f(x)$ defined for the interval $-L \le x \le L$,

and $f(x) = 0$ otherwise. Define a periodic function $\tilde{f}(x)$ of period =2L, i.e. $\tilde{f}(x) = \tilde{f}(x+2L)$

and $\tilde{f}(x) = f(x)$ for $-L < x < L$

Then, the Complex Fourier series for $\tilde{f}(x)$ is given by:

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x} \text{.........(1) where } c_n = \frac{1}{2L}\int_{-L}^{L} \tilde{f}(x) e^{-i\frac{n\pi}{L}x} dx = \frac{1}{2L}\int_{-\infty}^{\infty} f(x) e^{-i\frac{n\pi}{L}x} dx \text{......(2)}$$

Define $$F(i\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \text{.....................(3)}$$

Sub. (3) into (2) $\Rightarrow 2L \cdot c_n = F(i\frac{n\pi}{L}) = F(in\omega_0)$ where $\omega_0 = \frac{\pi}{L}$,

$$\Rightarrow \quad \tilde{f}(x) = \sum_{n=-\infty}^{\infty} \frac{F(in\omega_0)}{2L} e^{in\omega_0 x} = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(in\omega_0) e^{in\omega_0 x} \text{.......(4)}$$

As $L \to \infty$, $\tilde{f}(x)$ approaches $f(x)$ and consequently (4) becomes:

$$f(x) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(in\omega_0) e^{in\omega_0 x} \text{...............(5)}$$

Also, since $\omega_0 = \frac{\pi}{L}$, $\omega_0 \to 0$ as $L \to \infty$, let $\omega_0 = \Delta\omega$, and $n\omega = \omega_n$

thus, the summation in (5) becomes an integral:

$$\Rightarrow \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega) e^{i\omega x} d\omega \text{...............(6)}$$

From (3), $F(i\omega)$ is the so-called Fourier transform of a function $f(x)$.

Fourier transform: An example

Find the Fourier transform of $f(x) = \begin{cases} k, & \text{for } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

$$F(i\omega) = \mathcal{F}\left\{ f(x) \right\} = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{0}^{a} k e^{-i\omega x} dx = \frac{k}{-i\omega}(e^{-i\omega a} - 1) = \frac{i \cdot k}{\omega}(e^{-i\omega a} - 1)$$

Note that $F(i\omega)$ is in general a complex-value function.

Some useful Properties of Fourier transform

1. Linearity: $\mathcal{F}\{af(x)+bg(x)\} = \int_{-\infty}^{\infty}[af(x)+bg(x)]e^{-i\omega x}dx = a\mathcal{F}\{f(x)\}+b\mathcal{F}\{g(x)\}$

2. Time-shifting: $\mathcal{F}\{f(x-x_0)\} = e^{-i\omega x_0}\mathcal{F}\{f(x)\}$

3. Frequency-shifting: $\mathcal{F}\{e^{i\omega_0 x}f(x)\} = F(i(\omega-\omega_0))$

4. Time and Frequency Scaling: $\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F(\frac{i\omega}{a})$

Some useful Properties of Fourier transform (cont'd)

5. Differentiation in Time: If $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx = f(x)e^{-i\omega x}\Big]_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = i\omega\mathcal{F}\{f(x)\}$$

and $\mathcal{F}\{f''(x)\} = -\omega^2\mathcal{F}\{f(x)\}$

6. Integration: $\mathcal{F}\left\{\int_{-\infty}^{x} f(x^*)dx^*\right\} = \dfrac{1}{i\omega}F(i\omega) + \pi F(0)\delta(\omega)$

7. Differentiation in Frequency: $\mathcal{F}\{xf(x)\} = i\dfrac{dF(i\omega)}{d\omega}$

8. Convolution: Let the convolution $f * g$ of functions f and g be defined by:

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

then:

$$\mathcal{F}\{(f * g)(x)\} = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}$$

Fourier Transform of Periodic functions

The concept of Fourier transform also applies to periodic functions as well.

Consider $F(i\omega) = 2\pi\delta(\omega - \omega_0) \Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega x}d\omega = e^{i\omega_0 x}$

Thus, for $F(i\omega) = \sum_{n=-\infty}^{\infty} c_n \cdot 2\pi\delta(\omega - n\omega_0), \quad f(x) = \mathcal{F}^{-1}\{F(i\omega)\} = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$

or

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}\right\} = \sum_{n=-\infty}^{\infty} c_n \cdot 2\pi\delta(\omega - n\omega_0)................(*)$$

For any period function $f(x)$, say of period $= 2L$, since we can express $f(x)$ as a Complex Fourier series: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x} = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$ where $\omega_0 = \frac{\pi}{L}$

From (*), the Fourier transform of this periodic function $f(x)$ is given by:

$$F(i\omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}\right\} = \sum_{n=-\infty}^{\infty} c_n \cdot 2\pi\delta(\omega - n\omega_0)$$

where c_n's are the coefficients of the Complex Fourier series of $f(x)$.