Fourier Series and Applications

- Functions expansion is done to understand them better in powers of x etc.
- Many important problems involving partial differential equations can be solved, provided a given function can be expressed as an infinite sum of sines and cosines.
- In this section, we will see how functions can be expanded having discontinuities also. Applications are in rotating machines, Sound waves, heart Beats.
- These trigonometric series are called **Fourier series**, and are somewhat analogous to Taylor series, in that both types of series provide a means of *expressing complicated functions* in terms of certain familiar elementary functions.

Broad Use of Fourier Series

- Fourier series is used as a means of solving certain problems in partial differential equations.
- However, Fourier series have much wider application in science and engineering, and in general are valuable tools in the investigations of periodic phenomena.
- For example, a basic problem in spectral analysis is to resolve an incoming signal into its harmonic components, which amounts to constructing its Fourier series representation.
- In some frequency ranges the separate terms correspond to different colors or to different audible tones.
- **The magnitude of the coefficient determines the amplitude of each component.**

Important formulas

- A t ratio of (n*90±θ) =±same ratio of θ when n is even,(The sign +or – is to be decided from the quadrant in which the angle (n*90±θ) lies). Ex: sin 570=sin(6x90+30)= - sin30=-1/2.
- A t ratio of (n*90±θ) =±co ratio of ratio of θ when n is odd. (The sign +or – is to be decided from the quadrant in which the angle (n*90±θ) lies).
- Tan315=tan (3x90+45)=-cot45=-1

$$\int uvdx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + u'''' v_5 - ....$$

- Where dash denotes differentiation and suffixes integration w r to x.

Important Formulas

$$2 SinxCosy = Sin (x + y) + Sin (x - y),$$

$$2 CosxSiny = Sin (x + y) - Sin (x - y)$$

$$2 CosxCosy = Cos (x + y) + Cos (x - y),$$

$$2 SinxSiny = Cos (x - y) - Cos (x + y),$$

$$Sin \frac{\pi}{2} = \sin \frac{5\pi}{2} = ... = 1, \sin \frac{3\pi}{2} = \sin \frac{7\pi}{2} = ... = -1$$

$$\cos 2\pi = \cos 4\pi = ... = 1, \sin n\pi = 0, \cos n\pi = (-1)^n,$$

$$\cos \pi = \cos 3\pi = \cos 5\pi = -1$$

$$\sin \left(n + \frac{1}{2} \right) \pi = (-1)^n, \cos \left(n + \frac{1}{2} \right) \pi = 0$$

$$n = \text{int}$$

Important Formulas

$$\int_{\alpha}^{\alpha+2\pi} \cos nx dx = \left|\frac{\sin nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0,$$

$$\int_{\alpha}^{\alpha+2\pi} \sin nx dx = -\left|\frac{\cos nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0,$$

$$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} \left(\cos(m+n)x + \cos(m-n)x\right)dx$$

$$= \frac{1}{2}\left|\frac{\sin(m+n)}{(m+n)} + \frac{\sin(m-n)}{m-n}\right|_{\alpha}^{\alpha+2\pi} = 0,$$

Important Formulas

$$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx dx = \left|\frac{x}{2} + \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi, n \neq 0$$

$$\int_{\alpha}^{\alpha+2\pi} \sin^2 nx dx = \left|\frac{x}{2} - \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi, n \neq 0$$

$$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} \left(\sin(m+n)x + \sin(m-n)x\right)dx$$

$$= -\frac{1}{2}\left|\frac{\cos(m+n)}{(m+n)} + \frac{\cos(m-n)}{(m-n)}\right|_{\alpha}^{\alpha+2\pi} = 0, m \neq n$$

Important Formulas

$$\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx dx = \left| \frac{\sin^2(nx)}{(2n)} \right|_{\alpha}^{\alpha+2\pi} = 0, n \neq 0$$

$$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx dx = \frac{1}{2} \left| \frac{\sin(m-n)}{(m-n)} - \frac{\sin(m+n)}{m+n} \right|_{\alpha}^{\alpha+2\pi} = 0, m \neq n$$

Fourier Series Representation of Functions

- We begin with a series of the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

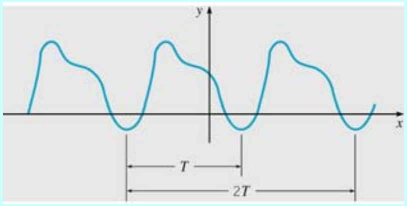
- On the set of points where this series converges, it defines a function *f* whose value at each point *x* is the sum of the series for that value of *x*.
- In this case the series is said to be the **Fourier series** of *f*.
- Our immediate goals are to determine what functions can be represented as a sum of Fourier series, and to find some means of computing the coefficients in the series corresponding to a given function.

Periodic Functions

- We first develop properties of $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where m is a positive integer.
- The first property is their periodic character.
- A function is **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if

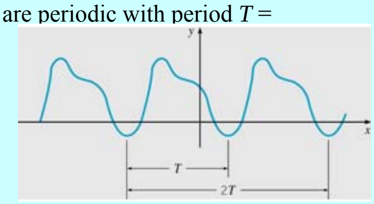
$$f(x + T) = f(x)\ ,$$

for all x. See the graph as below.



Periodicity of the Sine and Cosine Functions

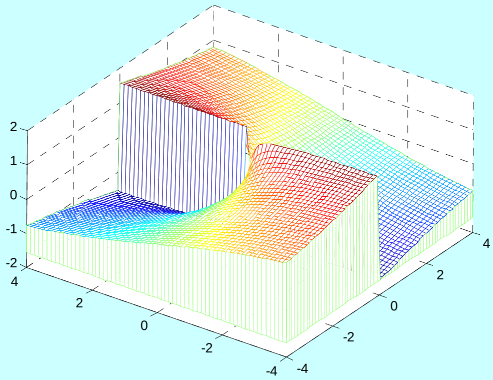
- For a periodic function of period T, $f(x + T) = f(x)$ for all x.
- Sin nx and Cos nx are periodic with period $2\pi/n$. Also $2T$ is also a period, and so is any multiple of T.
- The smallest value of T for which f is periodic is called the **fundamental period** of f.
- If f and g are two periodic functions with common period T, then fg and $c_1 f + c_2 g$ are also periodic with period T.
- In particular, $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ are periodic with period $T = 2L/m$.



$$f(x) = 2\sin x + \frac{1}{2}\cos 3x + \frac{1}{5}\sin 5x,$$

$$= 2.2\pi + \frac{1}{2}\frac{2\pi}{3} + \frac{1}{5}\frac{2\pi}{5}$$

Discontinuities



Orthogonality

- The standard inner product (u, v) of two real-valued functions u and v on the interval $\alpha \le x \le \beta$ is defined by

$$(u,v) = \int_{\alpha}^{\beta} u(x)v(x)dx$$

- The functions u and v are **orthogonal** on $\alpha \le x \le \beta$ if their inner product (u, v) is zero:

$$(u,v) = \int_{\alpha}^{\beta} u(x)v(x)dx = 0$$

- A set of functions is **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

Orthogonality of Sine and Cosine

- The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, \ldots$, form a mutually orthogonal set of functions on $-L \le x \le L$, with

  $\int_{-L}^{L} \cos(m\pi x/L) \cos(n\pi x/L)\, dx = L\, \delta_{m,n}$

  $\int_{-L}^{L} \cos(m\pi x/L) \sin(n\pi x/L)\, dx = 0$

  $\int_{-L}^{L} \sin(m\pi x/L) \sin(n\pi x/L)\, dx = L\, \delta_{m,n}$

  where $\delta_{m,n} = 1$ if m=n and $\delta_{m,n} = 0$ if m≠n

- These results can be obtained by direct integration;

Fourier Expansion (- π,+ π)

- Suppose the series converges, and call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

- Coefficients a_n, $n = 1, 2, \ldots$, can be found as follows.

$$\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx \cos nx dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nx \cos nx dx$$

- By orthogonality,

$$\int_{-\pi}^{\pi} f(x)\cos nx dx = a_n \int_{-\pi}^{\pi}\cos^2 nx dx = \pi a_n$$

Coefficient Formulas

- Thus from the previous slide we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 1, 2, \ldots$$

- To find the coefficient a_0, we have

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx dx = \pi a_0$$

- Thus the coefficients a_n are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 0, 1, 2, \ldots$$

- Similarly, the coefficients b_n are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \ldots$$

The Euler-Fourier Formula(- π,+ π)

- Thus the coefficients are given by the equations

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \ldots,$$

which known as the **Euler-Fourier formulas**.

- Note that these formulas depend only on the values of $f(x)$ in the interval $-\pi \le x \le \pi$. Since each term of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

is periodic with period 2pi, the series converges for all x when it converges in $-\pi \le x \le \pi$, and f is determined for all x by its values in $-\pi \le x \le \pi$.

$$f(x)=\frac{1}{2}(\pi-x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

- Find the F S. to represent x-x^2 from-pi to pi.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi-x)dx$$

$$\frac{1}{2\pi}\left[\pi x-\frac{x^2}{2}\right]_{-\pi}^{\pi}=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi-x)\cos nxd$$

$$a_n=\frac{1}{2\pi}\left[(\pi-x)\frac{\sin nx}{n}-(-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}[0]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi-x)\sin nxdx$$

$$=\frac{1}{2\pi}\left[(\pi-x)\frac{-\cos nx}{n}-(-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{(-1)^n}{n}$$

Using coeff. Just obtained

- We get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

Obtain the Fourier expansion of $f(x)=e^{-ax}$ in the interval $(-\pi, \pi)$.

EX

Ans

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}dx=\frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$=\frac{e^{a\pi}-e^{-a\pi}}{a\pi}=\frac{2\sinh a\pi}{a\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\cos nxdx$$

$$a_n=\frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx+n\sin nx\}\right]_{-\pi}^{\pi}$$

$$=\frac{2a}{\pi}\left[\frac{(-1)^n\sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \sin nx dx$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

Hence the F. S. Expansion

$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi \sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

$$f(0) = 1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

Example 1 function

- $f(x)=x^2, -\pi<x<\pi$, f is even so all $b_n=0$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi} x^2dx=\frac{2}{3}\pi^2,$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,$$

$$a_n=\frac{2}{\pi}\left[x^2\frac{\sin nx}{n}-\frac{2x}{n}\left(\frac{-\cos nx}{n}\right)+\frac{2}{n^2}\left(\frac{-\sin nx}{n}\right)\right]_0^{\pi}$$

$$a_n=0+\frac{4}{n^2}(-1)^n$$

Example 1 continues

- Hence the Fourier series is given by

$$f(x)=x^2=\frac{1}{2}\frac{2\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}\ldots\right],$$

$$x=0,$$

$$\frac{\pi^2}{12}=\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\ldots\right],$$

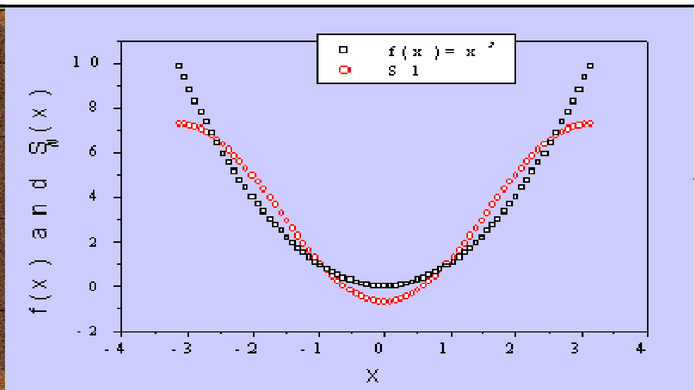
$$Lets define\ S_N(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{n=N} \frac{(-1)^n \cos nx}{n^2}$$

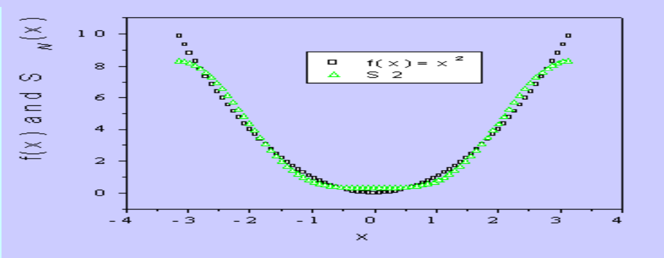
$$S_1(x) = \frac{\pi^2}{3} - 4\cos x$$

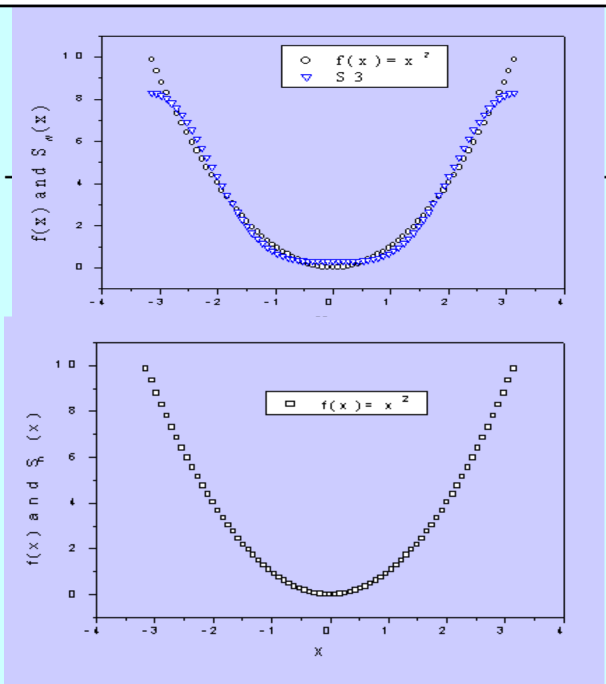
$$S_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x$$

$\ldots$

$$S_6(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x + \frac{1}{9}\cos 5x$$







Now coeff. in Fourier Expansion if (-L,L)

- Suppose the series converges, and call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

- The coefficients a_n, $n = 1, 2, \ldots$, can be found as follows.

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{n\pi x}{L}dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}dx$$

- By orthogonality,

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = a_n \int_{-L}^{L}\cos^2\frac{n\pi x}{L}dx = La_n$$

Coefficient Formulas

- Thus from the previous slide we have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \;\; n = 1,2,\ldots$$

- To find the coefficient a_0, we have

$$\int_{-L}^{L} f(x)dx = \frac{a_0}{2}\int_{-L}^{L}dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\frac{n\pi x}{L}dx = La_0$$

- Thus the coefficients a_n are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \;\; n = 0,1,2,\ldots$$

- Similarly, the coefficients b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \;\; n = 1,2,\ldots$$

The Euler-Fourier Formulas

- Thus the coefficients are given by the equations

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \;\; n = 0,1,2,\ldots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \;\; n = 1,2,\ldots,$$

which known as the **Euler-Fourier formulas**.

- Note that these formulas depend only on the values of $f(x)$ in the interval $-L \le x \le L$. Since each term of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

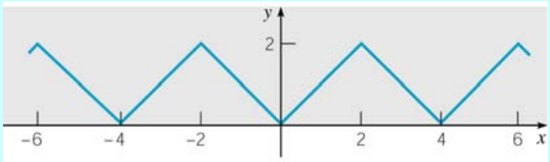
is periodic with period $2L$, the series converges for all x when it converges in $-L \le x \le L$, and f is determined for all x by its values in $-L \le x \le L$.

Example 2: Triangular Wave (1 of 3)

- Consider the function below.

$$f(x)=\begin{cases}-x, & -2\le x<0\\ x, & 0\le x<2\end{cases}, \qquad f(x+4)=f(x)$$

- This function represents a triangular wave, and is periodic with period $T = 4$. See graph of f below. In this case, $L = 2$.
- Assuming that f has a Fourier series representation, find the coefficients a_m and b_m.



Example 2: Coefficients (2 of 3)

- First, we find a_0:

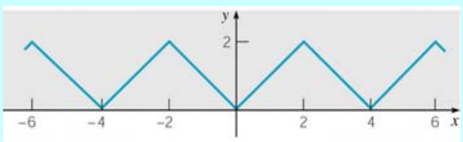
$$a_0=\frac{1}{2}\int_{-2}^{0}(-x)\,dx+\frac{1}{2}\int_{0}^{2}x\,dx=1+1=2$$

- Then for a_m, $m = 1, 2, \ldots$, we have

$$a_m=\frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{m\pi x}{2}dx+\frac{1}{2}\int_{0}^{2}x\cos\frac{m\pi x}{2}dx=\begin{cases}-8/(m\pi)^2, & m \text{ odd}\\ 0, & m \text{ even},\end{cases}$$

where we have used integration by parts.

- Similarly, it can be shown that $b_m = 0$, $m = 1, 2, \ldots$



Example 2: Fourier Expansion (3 of 3)

- Thus $b_m = 0$, $m = 1, 2, \ldots$, and

$$a_0 = 2,\ a_m = \begin{cases} -8/(m\pi)^2, & m \text{ odd} \\ 0, & m \text{ even}, \end{cases}$$

- Then

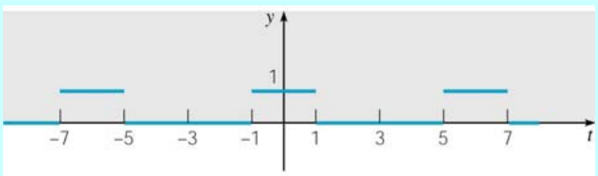
$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right) \\ &= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\frac{\cos(m\pi x/2)}{m^2} \\ &= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2} \end{aligned}$$

Example 3: Function (3 of 3)

- Consider the function below.

$$f(x) = \begin{cases} 0, & -3 < x < -1 \\ 1, & -1 < x < 1 \\ 0, & 1 < x < 3 \end{cases}, \qquad f(x+6) = f(x)$$

- This function is periodic with period $T = 6$. In this case, $L = 3$.
- Assuming that f has a Fourier series representation, find the coefficients a_n and b_n.



Example 3: Coefficients (3 of 3)

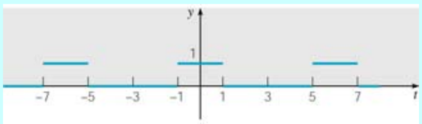
- First, we find a_0:

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_{-1}^{1} dx = \frac{2}{3}$$

- Using the Euler-Fourier formulas, we obtain

$$a_n = \frac{1}{3}\int_{-1}^{1} \cos\frac{n\pi x}{3}\,dx = \frac{1}{n\pi}\sin\frac{n\pi x}{3}\bigg|_{-1}^{1} = \frac{2}{n\pi}\sin\frac{n\pi}{3},\ n = 1, 2, \ldots,$$

$$b_n = \frac{1}{3}\int_{-1}^{1} \sin\frac{n\pi x}{3}\,dx = -\frac{1}{n\pi}\cos\frac{n\pi x}{3}\bigg|_{-1}^{1} = 0,\ n = 1, 2, \ldots,$$



Example 3: Fourier Expansion (3 of 3)

- Thus $b_n = 0$, $n = 1, 2, \ldots$, and

$$a_0 = \frac{2}{3},\ a_n = \frac{2}{n\pi}\sin\frac{n\pi}{3},\ n = 1, 2, \ldots$$

- Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{3}\cos\frac{n\pi x}{3}$$

$$= \frac{1}{3} - \frac{\sqrt{3}}{\pi}\left[\cos\left(\frac{\pi x}{3}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{3}\right) - \frac{1}{4}\cos\left(\frac{4\pi x}{3}\right) - \frac{1}{5}\cos\left(\frac{5\pi x}{3}\right) + \ldots\right]$$

Example 4: Triangular Wave

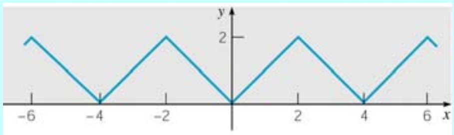
- Consider again the function from Example 1

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x < 2 \end{cases}, \qquad f(x+4) = f(x),$$

as graphed below, and its Fourier series representation

$$f(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x/2}{(2n-1)^2}$$

- We now examine speed of convergence by finding the number of terms needed so that the error is less than 0.01 for all *x*.



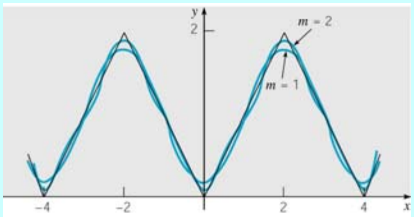
Example : Partial Sums

- The *m*th partial sum in the Fourier series is

$$s_m(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{m} \frac{\cos(2n-1)\pi x/2}{(2n-1)^2},$$

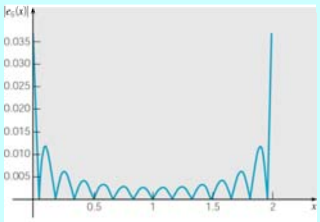
and can be used to approximate the function *f*.

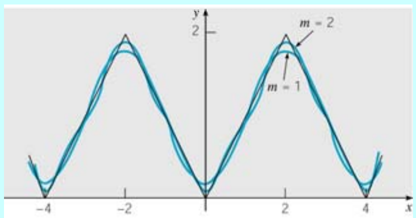
- The coefficients diminish as $(2n-1)^2$, so the series converges fairly rapidly. This is seen below in the graph of s_1, s_2, and *f*.



Example : Errors

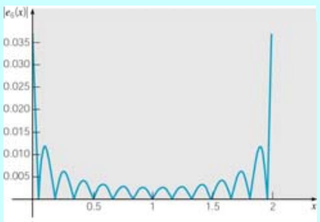
- To investigate the convergence in more detail, we consider the error function $e_m(x) = f(x) - s_m(x)$.
- Given below is a graph of $|e_6(x)|$ on $0 \le x \le 2$.
- Note that the error is greatest at $x = 0$ and $x = 2$, where the graph of $f(x)$ has corners.
- Similar graphs are obtained for other values of m.

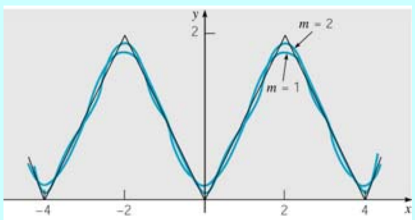




Example : Uniform Bound

- Since the maximum error occurs at $x = 0$ or $x = 2$, we obtain a uniform error bound for each m by evaluating $|e_m(x)|$ at one of these points.
- For example, $e_6(2) = 0.03370$, and hence $|e_6(x)| < 0.034$ on $0 \le x \le 2$, and consequently for all x.

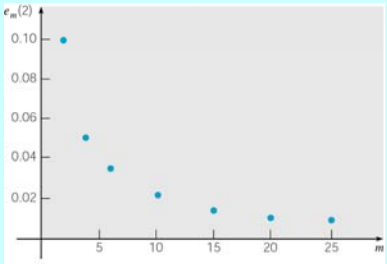




Example : Speed of Convergence

- The table below shows values of $|e_m(2)|$ for other values of m, and these data points are plotted below also.
- From this information, we can begin to estimate the number of terms that are needed to achieve a given level of accuracy.
- To guarantee that $|e_m(2)| \leq 0.01$, we need to choose $m = 21$.

m	e_m(2)
2	0.09937
4	0.05040
6	0.03370
10	0.02025
15	0.01350
20	0.01013
25	0.00810



At Last

- Thanks .