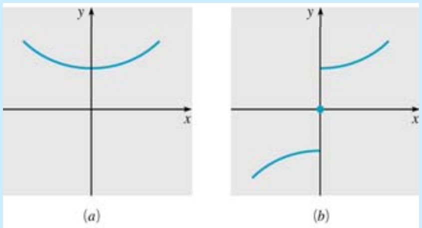
Even and Odd Functions

- Before looking at further examples of Fourier series it is useful to distinguish two classes of functions for which the Euler-Fourier formulas for the coefficients can be simplified.
- The two classes are even and odd functions, which are characterized geometrically by the property of symmetry with respect to the *y*-axis and the origin, respectively.

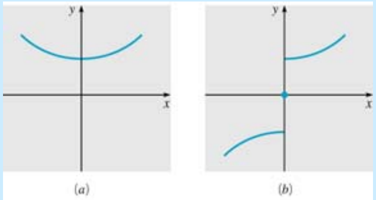
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$



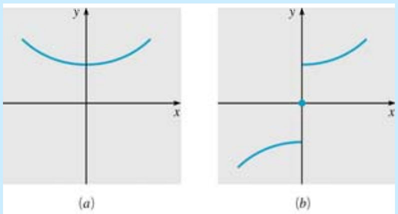
Definition of Even and Odd Functions

- Analytically, f is an **even function** if its domain contains the point $-x$ whenever it contains x, and if $f(-x) = f(x)$ for each x in the domain of f. See figure (a) below.
- The function f is an **odd function** if its domain contains the point $-x$ whenever it contains x, and if $f(-x) = -f(x)$ for each x in the domain of f. See figure (b) below.
- Note that $f(0) = 0$ for an odd function.
- Examples of even functions are 1, x^2, $\cos x$, $|x|$.
- Examples of odd functions are x, x^3, $\sin x$.



Arithmetic Properties

- The following arithmetic properties hold:
  - The sum (difference) of two even functions is even.
  - The product (quotient) of two even functions is even.
  - The sum (difference) of two odd functions is odd.
  - The product (quotient) of two odd functions is even.
- These properties can be verified directly from the definitions, see text for details.



Integral Properties

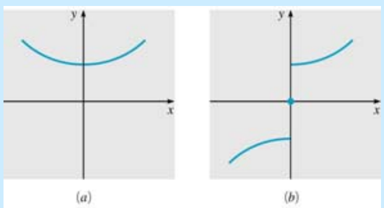
- If f is an even function, then

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

- If f is an odd function, then

$$\int_{-L}^{L} f(x)dx = 0$$

- These properties can be verified directly from the definitions, see text for details.



Cosine Series

- Suppose that f and f' are piecewise continuous on $[-L, L)$ and that f is an **even** periodic function with period $2L$.
- Then $f(x)\cos(n\pi x/L)$ is even and $f(x)\sin(n\pi x/L)$ is odd. Thus

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0,1,2,\ldots$$

$$b_n = 0, \; n = 1,2,\ldots$$

- Hence the Fourier series of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

- Thus the Fourier series of an even function consists only of the cosine terms (and constant term), and is called a **Fourier cosine series.**

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Sine Series

- Suppose that f and f' are piecewise continuous on $[-L, L)$ and that f is an **odd periodic function** with period $2L$.
- Then $f(x)\cos(n\pi x/L)$ is odd and $f(x)\sin(n\pi x/L)$ is even. Thus

$$a_n = 0, \quad n = 0,1,2,\ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1,2,\ldots$$

- It follows that the Fourier series of f is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

- Thus the Fourier series of an odd function consists only of the sine terms, and is called a **Fourier sine series.**

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Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).

- weget $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nxdx$$

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{n^3\pi}\left[1-(-1)^n\right]$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1-(-1)^n\right]\sin nx$$

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Obtain the cosine series over (0,pi) $f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases}$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} xdx + \int_{\pi/2}^{\pi}(\pi - x)dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nxdx + \int_{\pi/2}^{\pi}(\pi - x)\cos nxdx\right]$$

$$a_n = -\frac{2}{n^2\pi}\left[1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2,6,10,.....$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$$

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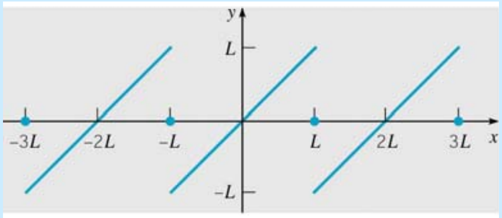
Example 1: Saw-tooth Wave (1 of 3)

- Consider the function below.

$$f(x) = \begin{cases} x, & -L < x < L \\ 0, & x = \pm L \end{cases}, \qquad f(x+2L) = f(x)$$

- This function represents a saw-tooth wave, and is periodic with period $T = 2L$. See graph of f below.
- Find the Fourier series representation for this function.



Example 1: Coefficients (2 of 3)

- Since f is an odd periodic function with period $2L$, we have

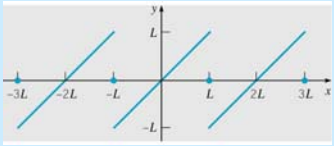
$$a_n = 0, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L} dx = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \left(\sin\frac{n\pi x}{L} - \frac{n\pi x}{L}\cos\frac{n\pi x}{L}\right)\Big|_0^L$$

$$= \frac{2L}{n\pi}(-1)^{n+1}, \quad n = 1, 2, \ldots$$

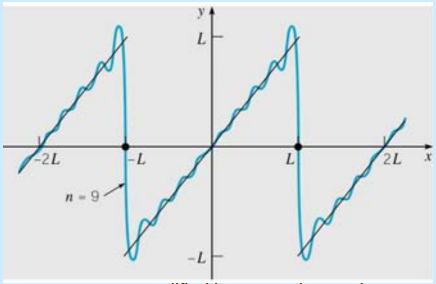
- It follows that the Fourier series of f is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$



Example 1: Graph of Partial Sum (3 of 3)

- The graphs of the partial sum $s_9(x)$ and f are given below.
- Observe that f is discontinuous at $x = \pm(2n+1)L$, and at these points the series converges to the average of the left and right limits (as given by Theorem 10.3.1), which is zero.
- The Gibbs phenomenon again occurs near the discontinuities.



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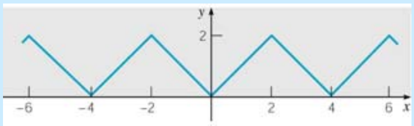
Even Extensions

- It is often useful to expand in a Fourier series of period $2L$ a function f originally defined only on $[0, L]$, as follows.
- Define a function g of period $2L$ so that

$$g(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L < x < 0 \end{cases}, \qquad g(x+2L) = g(x)$$

- The function g is the **even periodic extension** of f. Its Fourier series, which is a cosine series, represents f on $[0, L]$.
- For example, the even periodic extension of $f(x) = x$ on $[0, 2]$ is the triangular wave $g(x)$ given below.

$$g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x < 2 \end{cases}$$



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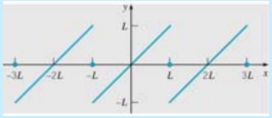
Odd Extensions

- As before, let f be a function defined only on $(0, L)$.
- Define a function h of period $2L$ so that

$$h(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, L \\ -f(-x) & -L < x < 0 \end{cases}, \qquad h(x+2L) = h(x)$$

- The function h is the **odd periodic extension** of f. Its Fourier series, which is a sine series, represents f on $(0, L)$.
- For example, the odd periodic extension of $f(x) = x$ on $[0, L)$ is the sawtooth wave $h(x)$ given below.

$$h(x) = \begin{cases} x, & -L < x < L \\ 0, & x = \pm L \end{cases}$$



General Extensions

- As before, let f be a function defined only on $[0, L]$.
- Define a function k of period $2L$ so that

$$k(x) = \begin{cases} f(x), & 0 \le x \le L \\ m(x), & -L < x < 0 \end{cases}, \qquad k(x+2L) = k(x)$$

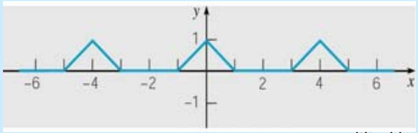
where $m(x)$ is a function defined in any way consistent with Theorem 10.3.1. For example, we may define $m(x) = 0$.
- The Fourier series for k involves both sine and cosine terms, and represents f on $[0, L]$, regardless of how $m(x)$ is defined.
- Thus there are infinitely many such series, all of which converge to f on $[0, L]$.

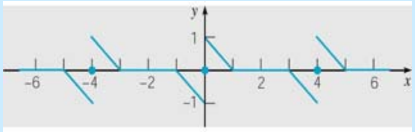
Example 2

- Consider the function below.

$$f(x) = \begin{cases} 1 - x, & 0 < x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$$

- As indicated previously, we can represent f either by a cosine series or a sine series on [0, 2]. Here, $L = 2$.
- The cosine series for f converges to the even periodic extension of f of period 4, and this graph is given below left.
- The sine series for f converges to the odd periodic extension of f of period 4, and this graph is given below right.





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