Partial Differential Equations: $u_{xx}+u_{yy}+u_{zz}=0$ Laplace Equation

$u_{tt}=c^2 u_{xx}$ Wave Eq^n.

Let us define $p=z_x$, $q=z_y$, $r=z_{xx}$, $s=z_{xy}$, $t=z_{yy}$. | $u_t=c^2 u_{xx}$ — heat Eq^n.

P.d.E can be obtained by elimination of arbitrary constants or elimination of arbitrary functions.

(a) Let $x^2+y^2+(z-c)^2=a^2$

$$2x+2(z-c)\frac{\partial z}{\partial x}=0 \quad \text{---(I)}$$

$$2y+2(z-c)\frac{\partial z}{\partial y}=0 \quad \text{---(II)}$$

Elimination of $(z-c)$ from both eq^{n}s

$$\Rightarrow xq-yp=0$$

(b) $z=f(x^2+y^2)$

$p=z_x=2xf'$, $z_y=2yf'=q$

$$\Rightarrow \frac{p}{q}=\frac{x}{y}$$

$$xq-yp=0$$

Some definitions: The solution $f(x,y,z,a,b)$ of a first order p.d.e with two arbitrary constants is called a complete integral. The solution obtained by eliminating a and b is called particular I. If we put $b=\phi(a)$, then find the envelop of family of surfaces $f[x,y,z,a,\phi(a)]=0$ & called genral integral.

Linear P.D.Eq: If it is in first degree in dependent variable and its partial derivatives.

Lagrange's First order Linear Equations: $Pp+Qq=R$, P,Q,R are $f(x,y,z)$

Solve $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ known as subsidiary equations & $\phi(u,v)=0$ or $u=f(v)$ $\Rightarrow u=f(v)$.

Note: If $\lambda P+\mu Q+\nu R=0$ Then $\lambda dx+\mu dy+\nu dz$ is integrable. λ,μ,ν are called Lagrange multipliers & can be taken as $\pm x, y, z$, $\pm 1/x$, $\pm 1/y$, $\pm 1/z$ or $dx\mp dy$, $dy\mp dz$, $dz\mp dx$ may be used.

Solve $(x^2-yz)p+(y^2-zx)q=(z^2-xy)$

Answer: $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, Performing

$$\Rightarrow \frac{dx-dy}{(x^2-yz-y^2+zx)}=\frac{dy-dz}{y^2-zx-z^2+xy}=\frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\Rightarrow \underbrace{\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}}_{\text{I}}=\frac{dz-dx}{(z-x)(x+y+z)} \quad \text{II}$$

$\Rightarrow \log(x-y)=\log(y-z)+\log c_1$

$\Rightarrow \frac{x-y}{y-z}=c_1$ first sol^n.

$\Rightarrow \frac{y-z}{z-x}=c_2$ second sol^n.

$$\phi\left(\frac{x-y}{y-z},\frac{y-z}{z-x}\right)=0$$

Solve $(mz-ny)p+(nx-lz)q=ly-mx$

Answer: $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

case (a) Using x,y,z as multipliers

$$\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$$

denominator is zero so. $xdx+ydy+zdz=0$ is integrable. So

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c \quad \text{---(I)}$$

(b) Similarly using l,m,n as multipliers, denominator is zero i.e

$l(mz-ny)+m(nx-lz)+n(ly-mx)$

$$\Rightarrow lx+my+nz=c_2 \quad \text{---(II)}$$

$$\phi(x^2+y^2+z^2,\ lx+my+nz)=0$$

Q. Solve $x(y-z)p+y(z-x)q=z(x-y)$

Answer

(i) $dx+dy+dz \neq xy-xz+yz-yx+zx-zy$

$\Rightarrow dx+dy+dz=0 \Rightarrow \boxed{x+y+z=c_1}$

(ii) $\frac{dx/x}{(y-z)}+\frac{dy/y}{(z-x)}+\frac{dz/z}{x-y} \Rightarrow$

$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0 \Rightarrow \boxed{xyz=c_2}$

Q. $(x^2-y^2-z^2)p+2xyq=2xz$

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$$

Last two gives $\boxed{y/z=c_1}$ — (1)

Now

$$\frac{x\,dx+y\,dy+z\,dz}{\cancel{x}(x^2+y^2+z^2)}=\frac{dz}{2\cancel{x}z}$$

$\Rightarrow \log x^2+y^2+z^2=\log z+\log c_2$

$\Rightarrow \phi\left(y/z,\ \frac{x^2+y^2+z^2}{z}\right)=0$

Q Solve

$$x(z^2-y^2)\frac{\partial z}{\partial x}+y(x^2-z^2)\frac{\partial z}{\partial y}=z(y^2-x^2)$$

Ans

Multipliers x, y & z and

$\frac{1}{x}, \frac{1}{y}$ & $\frac{1}{z}$

solution

$x^2+y^2+z^2=c_1$ — (1)

& $\log x+\log y+\log z=c_2$ — (ii)

$\phi(x^2+y^2+z^2,\ xyz)=0$

---

solve $px(z-2y^2)=(z-qy)(z-y^2-2x^3)$

Ans $\Rightarrow \{x(z-2y^2)\}p+\{y(z-y^2-2x^3)\}q=\{z(z-y^2-2x^3)\}$

$$\Rightarrow \frac{dx}{x(z-2y^2)}=\frac{dy}{y(z-y^2-2x^3)}=\frac{dz}{z(z-y^2-2x^3)}$$

First solution is easy by taking last two relations.

For second solution: Take $\frac{dx}{x(z-2y^2)}=\frac{dy}{y(z-y^2-2x^3)}$

Put $y=az \Rightarrow \frac{dx}{x(z-2a^2z^2)}=\frac{dy}{z(z-a^2z^2-2x^3)}$

$$\Rightarrow \frac{dx}{x\cancel{z}(1-2a^2z)}=\frac{dy}{\cancel{z}\ (\text{do})}$$

$(x\,dz-z\,dx)-a^2(2xz\,dz-z^2\,dx)+2x^3\,dx=0$

$$\Rightarrow \frac{x\,dz-z\,dx}{x^2}-a^2\frac{(2xz\,dz-z^2\,dx)}{x^2}+2x\,dx=0$$

$$\Rightarrow \frac{z}{x}-a^2\frac{z^2}{x}+x^2=b \text{ Answer.}$$

Partial differential equations: non linear

Forms (i) $f(p,q)=0$ (ii) $f(z,p,q)=0$ (iii) $f(x,p)=F(y,q)$ ~~(iv) z=pq~~ +
(iv) $z=px+qy+f(pq)$ (v) Charpit's Method

Form 1: $f(p,q)=0$ complete solution is given by $z=ax+by+c$ where we put $p=a$ and b is found by $f(a,b)=0$.

Example solve $p^2+q^2=1 \Rightarrow f(a,b)=a^2+b^2=1 \Rightarrow b=\sqrt{1-a^2}$, $p=a$
so solution is $z=ax+\sqrt{1-a^2}\,y+c$.

Example solve $x^2p^2+y^2q^2=z^2$. For this $\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2+\left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2=1$ --- (1)

Let $\frac{dx}{x}=du$, $\frac{dy}{y}=dv$, $\frac{dz}{z}=dw \Rightarrow u=\log x$, $v=\log y$, $w=\log z$.

From equation (1) we get $\left(\frac{dw}{du}\right)^2+\left(\frac{dw}{dv}\right)^2=1 \Rightarrow P^2+Q^2=1$ whose solution is by above form $\Rightarrow w=au+bv+c \Rightarrow au+\sqrt{1-a^2}\,v+c$ Ans.

Form 2: In this form we assum $z=\phi(y+ax)=\phi(u) \Rightarrow z_x=p=a\phi'(y+ax)$
$=a\frac{dz}{du}$ — (i)

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}$ and $z_y=q=\frac{\partial z}{\partial y}=\phi'(y+ax)=\frac{dz}{du}$ — (ii)

We substitute these values in given differential equation.

Solve $z^2=1+p^2+q^2 \Rightarrow z^2=1+a^2\left(\frac{dz}{du}\right)^2+\left(\frac{dz}{du}\right)^2 \Rightarrow z^2-1=(a^2-1)\left(\frac{dz}{du}\right)^2$

$$\Rightarrow \frac{dz}{du}=(a^2-1)^{1/2}\sqrt{z^2-1} \Rightarrow \frac{dz}{(z^2-1)^{1/2}}=(a^2-1)^{-1/2}du+c \Rightarrow \cosh^{-1}z=Mu+c$$

Solve $z^2(p^2x^2+q^2)=1$ Let we write $z^2\left[\frac{\partial z}{\left(\frac{\partial x}{x}\right)}\right]^2+\left(\frac{\partial z}{\partial y}\right)^2=1$

change $\frac{\partial x}{x}=dX \Rightarrow \log x=X \Rightarrow z^2\left\{\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}=1$. solve as above.

Form 3: Method put $f_1(x,p)=f_2(y,q)=a$ put $dz=p\,dx+q\,dy$ and integrate.

Example: $q=xyp^2 \Rightarrow \frac{q}{y}=xp^2=a$ (let) $\Rightarrow p=\sqrt{\frac{a}{x}}$, $q=ay$

so $dz=\sqrt{a/x}\,dx+ay\,dy \Rightarrow z=2\sqrt{ax}+(1/2)ay^2+c$ Ans.

Form 4: $z=px+qy+f(p,q)$, put $p=a$ & $b=q$ &
$z=ax+by+f(a,b)$ gives solution

Charpit's Method (1784): General method for finding solution of n.L.PDE of first order $f(x,y,z,p,q)=0$ --- (1)

Here we solve subsidiary eqns $\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dz}{-pf_p-f_q}=\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}$

solve these equations to get p and q and put in * along with (1)
$dz=p\,dx+q\,dy$ & solve.

Form $f(x,y,z,p,q)=0$

Example: charpits method: Find complete integral $z^2 = pqxy$

Answer: subsidiary equations are $\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-2pqxy} = \frac{dp}{pqy-2pz} = \frac{dq}{pqx-2qz}$

Perming $\frac{x\,dp + p\,dx}{-2pxz} = \frac{y\,dq + q\,dy}{-2qyz} \Rightarrow \frac{d(px)}{px} = \frac{d(qy)}{qy} \Rightarrow \log(px) = \log(qy) + \log c$

$\Rightarrow px/qy = c^2$ which is first solution ——— (I)

solving (I) and $z^2 = pqxy$ --- (II) $\rightarrow p = \frac{cz}{x}$ & $q = \frac{z}{cy}$

Hence $dz = p\,dx + q\,dy \Rightarrow dz = \frac{cz}{x}dx + \frac{z}{cy}dy$

$\Rightarrow \frac{dz}{z} = \frac{c}{x}dx + \frac{1}{cy}dy \Rightarrow \boxed{z = a x^c y^{1/c}}$ Answer.

Example solve $p^2 + 2z + qy + 2y^2 = 0$ (I) Let $f(x,y,z,p,q) = p^2 + 2z + qy + 2y^2 = 0$

subsidiary equations $\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2+qy)} = \frac{dp}{2p} = \frac{dq}{4y+3q}$

Taking $\frac{dx}{-2p} = \frac{dp}{2p} \Rightarrow p = -x + a$ --- (II) Now from (I) we get

$q = \frac{1}{y}[-2z - 2y^2 - (c-x)^2] \Rightarrow \because dz = (c-x)dx - \frac{1}{y}[2z + 2y^2 + (c-x)^2]dy$

multiplying both sides by $2y^2 \Rightarrow$

$2y^2 dz + 4yz\,dy = 2y^2(c-x)\,dx - 4y^3 dy - 2y(c-x)^2 dy$

Integrating we get $2zy^2 = -[y^2(c-x)^2 + y^4] + b$. soln.

Some General Problems:

(1) Solve $(x-y)(px-qy) = (p-q)^2$. (Hint $x+y=u$, $xy=v$)

(2) $z^2(p^2+q^2) = x^2+y^2$ (Hint put $z^2 = Z$

(3) $p^2x^2 + q^2y^2 = z$ — (Hint $x = e^X$, $y = e^Y$, $z^{1/2} = Z$)

(4) $(p^2+q^2)y = qz$ charpits Method.

Solve $p^2+q^2 = z^2(x+y) \Rightarrow \left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x+y$,

$\left(\frac{1}{z}z_x\right)^2 + \left(\frac{1}{z}z_y\right)^2 = x+y \Rightarrow \left(\frac{\partial z/z}{\partial x}\right)^2 + \left(\frac{\partial z/z}{\partial y}\right)^2 = x+y$ | Take $\frac{\partial z}{z} = d\mathfrak{z}$ $\Rightarrow \log z = \mathfrak{z}$

$\Rightarrow \left(\frac{\partial \mathfrak{z}}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{z}}{\partial y}\right)^2 = x+y \Rightarrow P^2 + Q^2 = x+y$

Let $P^2 - x = y - Q^2 = a \Rightarrow P = \sqrt{a+x}$, $Q = \sqrt{y-a}$

$d\mathfrak{z} = \frac{\partial \mathfrak{z}}{\partial x}dx + \frac{\partial \mathfrak{z}}{\partial y}dy \Rightarrow d\mathfrak{z} = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy$

$\Rightarrow \mathfrak{z} = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + C$

$\Rightarrow \log z =$ Ans —

Applications of partial differential equations: Seperations of Variable method.

Accoustic wave, water wave, electromagnetic wave

Seismic wave, Laplace equations, Vibrating Membranes, Transmission Lines equation.

Q1 solve $z_{xx} + z_y - 2z_x = 0$: Let $z = X(x).Y(y) \Rightarrow Y\frac{\partial^2 X}{\partial x^2} - 2\frac{\partial X}{\partial x} = -X\frac{\partial Y}{\partial y}$

Ans: $\frac{\partial^2 X/\partial x^2 - 2\partial X/\partial x}{X} = -\frac{\partial Y/\partial y}{Y} \Rightarrow \frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = k$ (let)

$\Rightarrow X'' - 2X' - kX = 0$ ---- (I) and $Y' + kY = 0$ ---- (II)

$\Rightarrow m^2 - 2m - k = 0$

$\Rightarrow m_1 = 1 + \sqrt{1+k}, m_2 = 1 - \sqrt{1+k}$

So $\boxed{X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}}$

$\frac{dY}{dy} = -kY \Rightarrow \frac{dY}{Y} = -k\,dy$

$Y/c = -ky \Rightarrow \boxed{Y = ce^{-ky}}$

Hence $z = \{c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\}\{ce^{-ky}\}$ Ans

Q.2 Solve $3u_x + 2u_y = 0$, $u(x,0) = 4e^{-x}$ i.e Boundary Conditions.

Answer Let $U = X(x).Y(y) \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial X}{\partial x}.Y$ & $\frac{\partial U}{\partial y} = X\frac{\partial Y}{\partial y}$, Put in Question

$3Y\frac{\partial X}{\partial x} + 2X\frac{\partial Y}{\partial y} = 0 \Rightarrow \frac{3}{X}\frac{\partial X}{\partial x} = -\frac{2}{Y}\frac{\partial Y}{\partial y} = k \Rightarrow \frac{3}{X}\frac{\partial X}{\partial x} = k$ ---(I) & $\frac{2}{Y}\frac{\partial Y}{\partial y} = -k$ ---(II)

Now equation (I) and (II) gives

$3\frac{\partial X}{X} = k\partial x \Rightarrow 3\log X = kx + \log C$

$\Rightarrow \boxed{X = ce^{kx/3}}$

From 2:

$2\log Y = -ky + \log c$

$\boxed{Y = ce^{-ky/2}}$

So $U = c\, e^{-ky/2} . e^{+kx/3}$ Applying condition (Boundys)

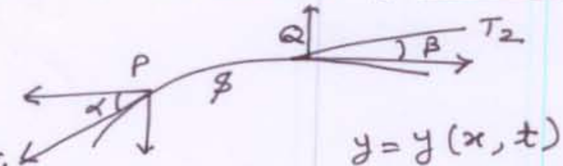
we get $4e^{-x} = ce^{-k.0/2} e^{kx/3} \Rightarrow c = 4e^{-x}/e^{kx/3}$ Ans.

Two Very Important PDE (I) Wave Eqn (II) Heat eqn

The solution of these are required w.r. to given boundary conditions which can be obtained by seperation of variable method and boundary conditions such as

(1) Dirichlets Type	(2) Newmann's Type	(3) Mixed Type.
$u(x,t) = u(0,t) = f_1(t)$ $u(L,t) = f_2 t$ or $y(0,0) = f_1 = y(x,l)$ $y(0\,l) = f_2$	$y = f(x)$ at $t = 0$ and $\frac{\partial y}{\partial t} = 0$, $t = 0$ Initial Condition $u(x,t) = f(x)$	$y = 0$, $t = 0$ $\frac{\partial y}{\partial t} = g(x)$, $t = 0$

Transverse vibration of elastic string



$y = y(x,t)$

$s \to$ length, $m \to$ mass per unit length

$y \to$ displacement, $\ddot{y} \to$ acceleration

Equations of motion will be

let $T = T_2 \cos\beta - T_1 \cos\alpha = 0$ ----(I)

$T_2 \sin\beta - T_1 \sin\alpha = F = m.\Delta s \dfrac{\partial^2 y}{\partial t^2}$ --(II)

From (I) and (II)

$$\frac{T_2 \sin\beta - T_1 \sin\alpha}{T} = \left(\frac{m\Delta s}{T}\right)\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\beta} = RHS,\ \text{Now as } \Delta s = \Delta x$$

$$\Rightarrow \frac{d^2 y}{dt^2} = \left(\frac{T}{m\Delta x}\right)\left\{\frac{\partial y}{\partial x}\Big|_{x+\Delta x} - \frac{\partial y}{\partial x}\Big|_{x}\right.$$

as $\tan\alpha = \partial y/\partial x$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2} \quad \text{as } \Delta x \to 0$$

$$\boxed{\partial^2 y/\partial t^2 = c^2\ \partial^2 y/\partial x^2} \quad \text{—(III)}$$

Bound Cond's (a) $y=0,\ x=0$ (b) $y=0,\ x=l$

may be (c) $y=f(x),\ t=0$ (d) $\dfrac{\partial y}{\partial t}=0,\ t=0$

solution by seperation of variables:

Let $y = X(x)Y(y) \Rightarrow X\dfrac{d^2T}{dt^2} = Tc^2\dfrac{d^2X}{dx^2}$

$$\Rightarrow \frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \text{ (say)}$$

$$\Rightarrow \ddot{T} = -k^2c^2T \quad \& \quad \ddot{X} = -k^2X$$

$$\Rightarrow T = A\cos ckt + B\sin ckt$$

$X = C\cos kx + D\sin kx$, So

$$y = (A\cos ckt + B\sin ckt)(C\cos kx + D\sin kx) \quad \text{(A)}$$

Applying Boun cond^n (a) $C=0$,

" " " (b) $\sin kl = 0 \Rightarrow k = \dfrac{n\pi}{l}$

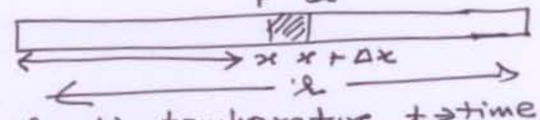
$$\Rightarrow y = \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \quad \text{(B)}$$

Apply (c) we get $A_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{\pi x n}{l}\,dx$

" (d) B_n's $= 0$ using orthogo of sin f^{n}s.

so eq^n **B** will give solution if $f(x)$ is known.

Heat Flow in a bar One Dim!



$u = u(x,t) =$ temperature, $t \to$ time

$A =$ Area of PQ, $\Delta t \to$ small time

Heat flow across the section at P in time $\Delta t = -kA\left(\dfrac{\partial u}{\partial x}\right)\Delta t$

Similarly at Q $\Rightarrow -kA\left(\dfrac{\partial u}{\partial x}\right)\Big|_{x+\Delta x}.\Delta t$

$$\text{Heat Retaind PQ} = -kA\left\{\frac{\partial u}{\partial x}\Big|_{x} - \frac{\partial u}{\partial x}\Big|_{x+\Delta x}\right\}\Delta t \quad \text{— (I)}$$

This heat will raise temperature of element by Δu temperature.

$$Q = (\rho A\Delta x)\,s.\Delta u \quad \text{— (II)}$$

(I)=(II) $\Rightarrow kA\{\ \} = \rho A.s\,\Delta x\Delta u$ —(III)

Dividing both sides by $\Delta x.\Delta t$ and $\Delta x \to 0$, $\Delta t \to 0$

$$\frac{kA}{\Delta x}\left\{\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_{x}\right\}\frac{\Delta t}{\Delta t} = \rho A\frac{s\,\Delta x\Delta u}{\Delta x\,\Delta t}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \left(\frac{\rho s}{k}\right)\frac{\partial u}{\partial t},\ \rho = \text{density},\ k = \text{conductibity}$$

$$\Rightarrow c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \to \text{Heat Equation.}$$

Bound cond^{n}s: (a) $x=0,\ u=0$, (c) $t=0$, $u=f(x)$

(b) $x=l,\ u=0$

solution by seperations of variables:

Let $U = X(x)T(t) \Rightarrow c^2T\dfrac{d^2X}{dx^2} = X\dfrac{dT}{dt}$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{dT}{dt} = -k^2$$

$$\frac{d^2X}{dx^2} + k^2X = 0,\ \frac{dT}{dt} + c^2k^2T = 0$$

$$\Rightarrow U = XT = (A\cos kx + B\sin kx)\,e^{-c^2k^2t} \quad \text{— (A)}$$

Applying Boundary Conditions (a) we get $A=0$, App B.C (b) we get $k = n\pi/l$ $(n \in I)$

$$\Rightarrow U = \sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{l}.\,e^{-t\left(\frac{n\pi c}{l}\right)^2} \quad \text{(B)}$$

Applying $t=0,\ u=f(x)$ & using orthogonality of sin f^n

$$f(x) = \sum B_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Q1. A tightly stretched string (l) with fixed ends is initially in equilibrium. It is set vibrating by giving each point a velocity $v_0 \sin^3 \frac{\pi x}{l}$. Find displacement $y(x,t)$.

Ans We know wave eqn $y_{tt} = c^2 y_{xx}$ —(I)

Boundary condns $y(x=0,t)=0$ —2(a)

$y(x, t=0)=0$ —(III), $y(x=l,t)=0$ —2(b)

$\left.\frac{\partial y}{\partial t}\right)_{t=0} = v_0\sin^3\frac{x\pi}{l}$ —(IV)

Genral solution of the equation I is

$y(x,t) = (c_1\cos kx + c_2 \sin kx)(c_3\cos kct + c_4\sin kct)$

Apply 2(a) $\Rightarrow 0 = c_1(c_3\cos ckt + c_4 \sin ckt)$

$\Rightarrow c_1=0$, Now apply 2(b)

$\Rightarrow k = n\pi/l$ for all t.

So genral solution reduces to

$y(x,t) = c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi c}{l}t + c_4\sin\right)$

applying (III) we get $c_3=0$

$y = \sum b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi c}{l}t$

$\frac{\partial y}{\partial t} = \sum b_n\left\{\sin\frac{n\pi x}{l}\right\}\frac{cn\pi}{l}\cos\frac{cn\pi t}{l}$

apply (4)

$v_0\sin^3\frac{\pi x}{l} = \sum \frac{n\pi c}{l} b_n \sin\frac{n\pi x}{l}$

{using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$}

$\frac{v_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = \sum b_n \frac{cn\pi}{l}\sin\frac{n\pi x}{l}$

$= \frac{c\pi}{l}b_1\sin\frac{\pi x}{l} + \frac{2c\pi}{l}b_2\sin\frac{2\pi x}{l} + b_3\frac{3c\pi}{l}\sin\frac{3\pi x}{l}+\dots$

Compair both sides

$\Rightarrow b_1 = \frac{3lv_0}{4c\pi}$, $b_3 = -\frac{lv_0}{12c\pi}$, $b_2=b_4=b_6=0$

$\Rightarrow$

$$y = \frac{lv_0}{12c\pi}\left[9\sin\frac{\pi x}{l}\sin\frac{c\pi t}{l} - \sin\frac{3\pi x}{l}\sin\frac{3c\pi t}{l}\right]$$

Ans

stretched string

Q.2 string length l stretched. Motion started by displacing string in form $y = a\sin\frac{x\pi}{l}$ which is released at $t=0$. Find $y(x,t) = ?$

Ans: condns: $y(x,0)= a\sin(\pi x/l)$ —(5)

Initial transverse velocity $\left.\partial y/\partial t\right|_{t=0} = 0$ —(4)

$y(0,t)=0$ —(2)

$y(l,t)=0$ —(3)

Applying (2) $c_1=0$, Apply (3) $k=\frac{n\pi}{l}$

Applying cond (4)

$\left.\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2\sin\frac{n\pi}{l}x\right) c_4\cdot c\frac{n\pi}{l}$

$\Rightarrow c_4=0$, $c_2\neq0$ as it will lead to trivial solution

$\Rightarrow y(x,t) = c_2c_3\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$

applying condn (5).

$a\sin\frac{\pi x}{l} = c_2c_3\sin\frac{n\pi x}{l}$

$\Rightarrow c_2c_3=a$ & $n=1$

$\Rightarrow \boxed{y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}}$ soln A

Physical interpretation

$\lambda_n = cn\pi/l$ called eigenvalues corresponding to $y(x,t)=c_2c_3\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$ eigen functions.

Solution A is Sin wave $y = y_0\sin\frac{x\pi}{l}$ of wave length l, wave velocity c and amplitude $y_0 = a\cos\frac{\pi t}{l}$ which varies harmonically with time t. What ever t may be $y=0$ when $x=0, l, 2l, 3l$--- called nodes. A represents stationary sinewave of varying amplitudes whose frequency is $c/2l$.

Ex1: A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0°C$ and kept at that temperature. Find $u(x,t)$

Ans $\quad u_t = c^2 u_{xx}$ — (1)

$u(0,t)=0$, $u(l,t)=0$ — 2b, 2(a)

General sol^n

$u(x,t) = (c_1' \cos px + c_2' \sin px) \times e^{-p^2c^2t}$ — (3)

Applying 2(a) $\Rightarrow c_1' = 0$

— — — 2(b) $\Rightarrow p = n\pi/l$.

So 3 gives

$u_n = b_n e^{-(n^2c^2\pi^2 t/l^2)} \cdot \sin \frac{n\pi}{l} x$

Also given $u = u_0$ at $t=0$

$u_0 = b_n \sin \frac{n\pi x}{l} \Rightarrow$

$b_n = $ Ans

Ex2 Solve $u_t = \alpha^2 u_{xx}$ for conduction of heat along a rod without radiation s/t following conditions:

(i) $u \neq \infty$ for $t \to \infty$

(ii) $u_x = 0$ for $x=0, x=l$

(iii) $u = lx - x^2$ for $t=0$ between $x=0$ and $x=l$

Ans: Let $U = X(x)\,T(t)$

$X'' + k^2 X = 0$, $T' + k^2\alpha^2 T = 0$

$U = (c_1 \cos kx + c_2 \sin kx)\, c_3 e^{-k^2\alpha^2 t}$ — (A)

is valid if $-k^2$ is RHS.

If $k^2 = 0$, $X = c_7 x + c_8$, $T = c_9$

applying $U = (c_7 x + c_8)\cdot c_9$

applying (ii) we get $c_7 = 0$

$\boxed{U = c_8 c_9 = \alpha_0}$ — B

So

$U = \alpha_0$

From A

$u_x = (-c_1 \sin kx + c_2 \sin kx) \cdot k\, c_3 e^{-k^2\alpha^2 t}$

Applying (ii) we get

$c_2 = 0$, $\Rightarrow k = \frac{n\pi}{l}$

$U = \alpha_0 + \sum \alpha_n \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}}$

using the condition (iii) in above

$lx - x^2 = \alpha_0 + \sum \alpha_n \cos \frac{n\pi x}{l}$

This being expansion of $lx - x^2$ as a half range cosine in $(0,l)$.

$\alpha_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = l^2/6$, $\quad \alpha_n = \frac{2}{l}\int_0^l (lx-x^2)\cos\frac{n\pi x}{l}\,dx$

$= \frac{2}{l}\left\{ -\frac{l^3}{n^2\pi^2}(\cos n\pi + 1)\right\}$

$= -\frac{4l^2}{n^2\pi^2}$ when n even. otherwise 0

So taking $n = 2m$

$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right) e^{-(4m^2\pi^2\alpha^2 t/l)}$

Ans