Isolated Singularity, pole essential singularity, Branch point

Zero's of an analytic function: If $f(z)$ an analytic function is zero i.e. $f(z)=0$ at a point z_0 in R then z_0 is called a zero of $f(z)$. If $f(z)=0$ but $f'(z)\neq 0$ then zero is called simple zero. If $f(z_0)=f'(z_0)=f''(z_0)= \cdots = f^{n-1}(z_0)=0$, $f^{n}(z_0)\neq 0$. Then z_0 is called a zero of order n. For ex: $f(z)=e^{-1/z^2}=0$ $\Rightarrow z=0$ is a zero of order 2.

Singular Point: A point where $f(z)$ fails to be analytic is called singular point.

Isolated Singularity: If $z=z_0$ is a singularity of $f(z)$ s.t. $f(z)$ is analytic at each point in its neighbourhood, then $z=a$ is called an isolated singularity.

Example ① $f(z)=\tan z$ has isolated singularity at $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$ --- as $\tan z=0 \Rightarrow \tan z=\tan n\frac{\pi}{2}$, $n=1,3,5$ ---

(ii) $f(z)=\cot(\pi/z)=1/\tan(\pi/z) \Rightarrow \frac{\pi}{z}=n\pi$ or $\Rightarrow z=1/n$ $(n=1,2,3 ---)$

Thus $z=1, 1/2, 1/3$ -- are all isolated singularities as there are no other singularities in their neighbourhood.

Removable singularity: If a single valued function $f(z)$ is not defined at $z=a$ but $\lim_{z\to a} f(z)$ exist then $z=a$ is called removable singula. In such cases we define $f(z)$ at $z=a$ as equal to $\lim_{z=a} f(z)$. Example: $\lim_{z\to 0}\frac{\sin z}{z}=1$, $z=0$ R.S.

Pole: If in Laurent series expansion of $f(z)$ have only finite number of point given by $f(z)=\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\frac{a_{-3}}{(z-a)^3}+\cdots+\frac{a_{-n}}{(z-a)^n}$, $a_n\neq 0$

Then $z=a$ is called a pole of order n.

Vis a vis~ If $f(z)$ has a pole at $z=a \Rightarrow \lim_{z\to a} f(z)=\infty$

aliter: If we can find a positive integer n such that $\lim_{z\to z_0}(z-z_0)^n f(z)=A\neq 0$. Then $z=z_0$ is called a pole. If $n=1$, z_0 is called simple pole.

Example: (i) $f(z)=\frac{1}{(z-2)^3}$ has a pole of order 3 at $z=2$

(ii) $f(z)=\frac{\cot\pi z}{(z-a)^2} \Rightarrow \sin\pi z.(z-a)^2=0 \Rightarrow$ either $\sin\pi z=0$ or $(z-a)^2=0$ ... $z=a$ is a pole of order 2. and $\sin\pi z=\sin\pi n \Rightarrow z=n \Rightarrow z=\infty$ is limit point of these poles so $z=\infty$ is essential singularity.

Essential Singularity: A singularity which is not a pole or removable singularity. In $f(z)$ principal part of Laurent expansion has infinite terms.

Ex: (i) $e^{1/z}=1+1/z+\frac{1}{\lfloor 2\, z^2}+\frac{1}{\lfloor 3\, z^3}+\cdots$, $z=0$ is essential singularity.

(ii) $e^{1/z-2} \Rightarrow z=2$ is essential singularity.

Q: Find Laurent Series about indicated singularity

(a) $\frac{e^{2z}}{(z-1)^3}, z=1$ (b) $\frac{z-\sin z}{z^3}, z=0$ (c) $(z-3)\sin\frac{1}{(z+2)}, z=-2$

Ans: (a) Let $z-1=u \Rightarrow u+1=z$.

$$\frac{e^{2(u+1)}}{u^3} = e^2\frac{e^{2u}}{u^3} = \frac{e^2}{u^3}\left\{1+2u+\frac{(2u)^2}{\lfloor 2}+\frac{(2u)^3}{\lfloor 3}+\cdots\cdots\right\}$$

$$= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{(z-1)}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+\cdots$$

$\Rightarrow z=1$ is pole of order 3 or triple pole.

(b) $\frac{z-\sin z}{z^3} = \left(\frac{1}{\lfloor 3}-\frac{z^2}{\lfloor 5}+\frac{z^4}{\lfloor 7}-\cdots\right)$ $\Rightarrow z=0$ remov. sing.

(c) Let $z+2=u \Rightarrow z=u-2$

$\Rightarrow (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u}-\frac{1}{\lfloor 3}\frac{1}{u^3}+\frac{1}{\lfloor 5}\frac{1}{u^5}-\cdots\right\}$

$\Rightarrow 1-\frac{5}{z+2}-\frac{1}{6(z+2)^2}+\frac{5}{6(z+2)^3}+\cdots\cdots,$

$z=-2$ is an essential singularity.

Branch Point: A point $z=z_0$ is called a branch point of multiple valued function $f(z)$ if branches of $f(z)$ are interchanged when z described a closed path about z_0. Ex: $(z-3)^{1/3}=f(z)$, $z=3$ branch point

Evaluate $\int_C \frac{z-3}{z^2+2z+5}dz$ where C is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$

Answer: $z^2+2z+5=0 \Rightarrow z=\frac{-2\pm\sqrt{4-20}}{2}=-1\pm 2i$

(i) Both poles lie outside the circle $|z|=1$. Therefore using cauchy theorem $\int_C f(z)dz=0$.

(ii) Here only one pole $z=-1+2i$ lie inside the circle $C: |z+1-i|=2$. Therefore $f(z)$ is analytic within C except at this pole.

$\therefore \operatorname{Re} f(-1\pm 2i) = \lim_{z\to -1+2i}\frac{(z-3)(z+1-2i)}{(z+1-2i)(z+1+2i)} = \frac{-4+2i}{4i}$

$= i+1/2$ Ans.

Hence by residue theorem $\int_C f(z)dz = 2\pi i\{\operatorname{Res} f(-1+2i)\}$
$= \pi(i-2)$

(iii) Only $z=-1-2i$ lies inside circle $C: |z+1+i|=2$. Therefore

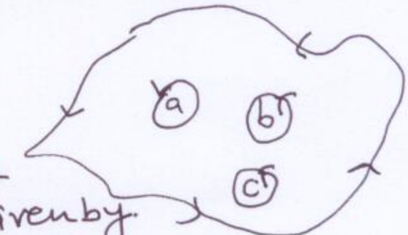
$\operatorname{Res} f(-1-2i) = \lim_{z\to -1-2i}\frac{(z+1+2i)(z-3)}{z^2+2z+5} = \lim\frac{z-3}{z+1-2i} = \frac{1}{2}-i$

$\int_C f(z)dz = 2\pi i\operatorname{Res} f(-1-2i) = \pi(2+i) = 2\pi i\left(\frac{1}{2}-i\right)$

Residue Theorem

Residue Theorem: If $f(z)$ is analytic single valued function inside and on a simple closed curve C except at a finiti number of points a, b, c inside C at which the residues are $a_{-1}, b_{-1}, c_{-1}, \cdots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\cdots)$$



Calculation of Residues:

(a) If $z=a$ is a simple pole then Laurent series expansion of $f(z)$ around pole is given by

$$f(z) = c_0 + c_1(z-a)+\cdots + a_{-1}(z-a)^{-1}$$

Multiplying with $(z-a)$ and taking limit as $(z-a)$ we get $\longrightarrow$ Formula

$$a_{-1} = \operatorname{Res} f(z)\Big|_{z=a} = \lim_{z\to a}\{(z-a)\, f z\}$$

(b) Similarly if $z=a$ is pole of order m then

$$a_{-1} = \frac{1}{\lfloor(m-1)}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}.$$

Example: Find Residue of $f(z) = z/\{(z-1)(z+1)^2\}$.

Answer Res at $z=1$, $a_{-1} = \lim_{z\to 1}\{\cancel{(z-1)}(z)/\cancel{(z-1)}(z+1)^2\} = 1/4$.

Res $z=-1$, $a_{-1} = \lim_{z\to -1}\frac{1}{\lfloor(2-1)}\frac{d}{dz}\{(z-(-1))^2\, z/(z-1)(z+1)^2\} = \frac{1}{\lfloor 1}\frac{d}{dz}\left\{\frac{z}{z-1}\right\} = -\frac{1}{4}$.

↓ pole order 2.

Example: Find the residue of $f(z) = (z^2-2z)/(z+1)^2(z^2+4)$.

Ans: Residue at $z=-1$ is given by

$$\lim_{z\to -1}\frac{1}{\lfloor 1}\frac{d}{dz}\left\{\frac{(z+1)^2(z^2-2z)}{(z+1)^2(z^2+4)}\right\} = \lim_{z\to -1}\left\{\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}\right\} = \frac{-14}{25}.$$

(ii) Residue at $z=2i$ is given by

$$\lim_{z\to 2i}\left\{\frac{(z-2i)(z^2-2z)}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}.$$

(iii) Residue at $z=-2i$ is given by

$$\lim_{z\to -2i}\left\{\frac{(z+2i)(z^2-2z)}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7+i}{25}$$

Example Find the residue $f(z) = e^z/\sin^2 z$ at all poles.

Answer: $f(z)$ has double poles at $z = m\pi$, $m = 0, \pm 1, \pm 2 \cdots$.

$$\lim_{z\to m\pi}\frac{1}{\lfloor 1}\frac{d}{dz}\left\{\frac{(z-m\pi)^2 e^z}{\sin^2 z}\right\} = \lim_{z\to m\pi}\left\{\frac{\sin^2 z\,(e^z(z-m\pi)^2+e^z\cdot 2(z-m\pi)) - 2(z-m\pi)^2 e^z\sin z\cdot\cos z}{\sin^2 z\cdot\sin^2 z}\right\}$$

$$= \lim_{z\to m\pi} e^z[(z-m\pi)^2\sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2\cos z]/\sin^3 z$$

Put $z - m\pi = u$

$$= e^{m\pi}\lim_{u\to 0}\frac{e^u[u^2\sin u + 2u\sin u - 2u^2\cos u]}{u^3}\,(u^3/\sin^3 u)$$

$$= e^{m\pi}\left\{\lim_{u\to 0}(e^u\sin u/u) + (2e^u\sin u/u^2) - (2e^u\cos u/u)\right\}\cdot 1 = e^{m\pi}$$

Ans.

Q1 Application Evaluati $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$ where c is the circle $|z| = 3$

Answer: $f(z)$ is analytic within the circle except $z=1$ which is double pole and $z=2$ simple pole.

(i) Residue $f(1) = \frac{1}{\lfloor 1} \left[\frac{d}{dz} \frac{(z-1)^2 (\sin \pi z^2 + \cos \pi z^2)}{(z-1)^2 (z-2)}\right]_{z=1}$

$$= \left[\frac{(z-2)\{2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2\} - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]_{z=1}$$

$= 2\pi + 1$

(ii) Res $f(2) = \lim_{z \to 2} \{(z-2) f(z)\} = \lim_{z \to 2} \left\{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2}\right\}$

$= 1$

By. residue theorem

$\int_C f(z) dz = 2\pi i \{\text{Res } f(1) + \text{Res } f(2)\} = 2\pi i \{2\pi + 1 + 1\}$ Ans.

Q.2 Evaluati $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$, Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$

Ans $\cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$.

$$I = \int_C \frac{\frac{1}{2}(z^3 + \frac{1}{z^3})}{5 - 2(z + \frac{1}{z})} \frac{dz}{iz} = -\frac{1}{2i} \int_C \frac{z^6 + 1}{z^3 (2z-1)(z-2)} dz$$

$= -\frac{1}{2i} \oint f(z) dz$ where c is unit circle $|z| = 1$, order $z = 0$ & $z = \frac{1}{2}$ lie inside circle

$\text{Res} f(1/2) = \lim_{z \to 1/2} \left\{\frac{\{z^6 + 1\}}{2z^3 (z-2)}\right\} = -65/42$

$\text{Res } f(0) = \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dz^{n-1}} [(z-0)^n f(z)]_{z=0}$ when $n = 3$

$= \frac{1}{2} \frac{d^2}{dz^2} \left\{(z^6 + 1) / (2z^2 - 5z + 2)\right\}_{z=0}$

$= 21/8$

$I = -\frac{1}{2i} \left\{(\text{Res}(1/2) + \text{Res}(0)) \times 2\pi i\right\}$

$= \pi/12$ Ans

Contour Integration